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Can TeVeS be a viable theory of gravity?

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ABSTRACT

Among modified gravitational theories, the Tensor–Vector–Scalar (TeVeS) occupies a special place – it is a covariant theory of gravity that produces the modified Newtonian dynamics (MOND) in the nonrelativistic weak field limit and explains the astrophysical data at scales larger than that of the Solar System, without the need of an excessive amount of invisible matter. We show that, in contrast to other modified theories, TeVeS is free from ghosts. These achievements make TeVeS (and its nonrelativistic limit) a viable theory of gravity. A speculative outlook on the emergence of TeVeS from a quantum theory is presented.

1. Introduction

The current accepted theory of gravity is Einstein's General Relativity (GR), which has been experimentally tested in the Solar System with great success. On the galactic and cosmological scales, however, the observed dynamics does not agree with the observed distribution of matter, when GR is taken as the theory of gravity. In order to make GR consistent with the observations on the galactic and cosmological scales, we have to postulate new invisible forms of energy, commonly referred to as dark matter and dark energy, which constitute the major part of the energy in the Universe. Neither of these dark elements has been observed by means other than their interaction with gravity. Since the postulation of such invisible elements may be a specious solution, we have to consider some other alternatives: GR may have to be amended.

We consider the Tensor–Vector–Scalar theory of gravity [1] (TeVeS) as an alternative to GR. TeVeS is a relativistic theory of gravity, which produces the modified Newtonian dynamics [2,3] (MOND) in the nonrelativistic weak field limit. According to the MOND paradigm, there exists an acceleration scale a_0 such that for accelerations smaller than a_0, Newton's second law is modified so that the gravitational force is proportional to the square of the particle's acceleration. It is remarkable that this simple proposal is so successful in explaining the galactic rotation curves [4], thus alleviating the need for dark matter on the galactic scales. For a review on MOND and TeVeS, see [5] and [6], respectively. Further on, TeVeS has been shown to be free of acausal propagation of perturbations and it is in agreement with solar system tests [1].

A priori, the missing mass could be composed of baryons in objects other than stars, such as brown dwarfs, Jupiter sized planets, or any kind of normal matter which is unseen presently. However, the experimental observations do not confirm the abundance of these objects [7]. The simplest explanation for the discrepancy between the dynamics and matter distribution is to postulate a new form of non-baryonic matter, the so-called dark matter, which does not interact with electromagnetic radiation. In order to explain the observed extra gravitational force, the abundance of dark matter has to be over five times greater than the observed amount of visible matter. The dark matter is traditionally split into hot dark matter and cold dark matter. Hot dark matter consists of particles that travel with ultrarelativistic velocities. The best candidate for the identity of hot dark matter is the neutrino, albeit the observed left-handed neutrinos with masses of few electron volts cannot constitute the bulk of dark matter. A right-handed neutrino could be a viable candidate for the role of dark matter, but such particles have not been detected so far. Cold dark matter is composed of massive slowly moving and weakly interacting particles. A number of such particles arise in particle physics models beyond the standard model [8]. These candidates have been studied

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extensively with results that are in agreement with experiments. However, these particles have not been detected so far. Moreover, the discovery of the accelerated expansion of the Universe [9] calls for another new form of invisible energy, known as dark energy. The dark energy provides most of the energy density in the Universe and it has to provide negative pressure. There are many proposals considering the explanation of dark energy [10], but no compelling candidate.

The alternative possibility for explaining the phenomena that are attributed to dark energy and dark matter is to revise the theory of gravity. There exist several kinds of modified or alternative gravitational theories, e.g., (Brans–Dicke) scalar tensor theories, Gauss–Bonnet gravities, $f(R)$ gravity, brane world models, conformal gravity, Poincaré gauge theories and many more. In this letter, we concentrate on TeVeS alone. TeVeS is a highly interesting theory of gravity, since it is a relativistic theory, it obeys the Einstein equivalence principle and produces the MOND phenomenology. On the other hand, in TeVeS, the gravitational vector and scalar fields are coupled to the metric of spacetime in a nonminimal way, which means that the local dynamics of the relativistic theory is involved and rich. The propagation of perturbations in the linearized theory has been studied in [11]. We study the full nonlinear theory using the Arnowitt–Deser–Misner (ADM) decomposition of the gravitational field [12] and the Hamiltonian formalism. We show that TeVeS is free from ghosts, which is a necessary condition for the consistency of the theory. Ghosts are notorious for causing instability in several theories, for example, in the renormalizable Weyl-like theories of gravity [13,14].

2. Fundamentals of TeVeS

TeVeS contains extra gravitational degrees of freedom, which are carried by a vector field A_μ and a scalar field ϕ. We emphasize that TeVeS involves two frames: the Bekenstein frame for the gravitational fields and a physical frame for the matter fields. The Bekenstein frame has the metric $\tilde{g}_{\mu\nu}$ with the connection $\tilde{\nabla}_\mu$. The action for all matter fields is written using a physical metric $g_{\mu\nu}$ with the connection ∇_μ, which is related to the three gravitational fields $\tilde{g}_{\mu\nu}$, A_μ and ϕ as

$$g_{\mu\nu} = e^{-2\phi}\tilde{g}_{\mu\nu} - 2\sinh(2\phi)A_\mu A_\nu. \tag{1}$$

The fact that all matter fields couple to the physical metric means that the Einstein equivalence principle is obeyed. The vector field is required to be timelike and normalized with respect to the Bekenstein metric,

$$A_\mu A^\mu = -1, \tag{2}$$

where the covariant index is raised with the Bekenstein metric, $A^\mu = \tilde{g}^{\mu\nu}A_\nu$. The inverse of the physical metric is obtained as

$$g^{\mu\nu} = e^{2\phi}\tilde{g}^{\mu\nu} + \frac{2\sinh(2\phi)e^{2\phi}}{e^{2\phi} - 2\sinh(2\phi)(A_\mu A^\mu + 1)}A^\mu A^\nu. \tag{3}$$

The action of the theory,

$$S = S_{\tilde{g}} + S_A + S_\phi + S_m, \tag{4}$$

consists of the actions for the metric $\tilde{g}_{\mu\nu}$, the vector field A_μ, the scalar field ϕ and matter, respectively. The action for $\tilde{g}_{\mu\nu}$ is defined as the standard Einstein–Hilbert action,

$$S_{\tilde{g}} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\tilde{R} + \frac{1}{8\pi G}\oint_{\partial\mathcal{M}} d^3x\sqrt{|\gamma|}\tilde{K}, \tag{5}$$

where G is the bare gravitational constant, $\tilde{g} = \det\tilde{g}_{\mu\nu}$, and $\tilde{R}$ is the scalar curvature defined by the connection $\tilde{\nabla}$ which is compatible with the metric $\tilde{g}_{\mu\nu}$. The surface integral over the boundary $\partial\mathcal{M}$ of the spacetime $\mathcal{M}$ is included so that only the variation of the metric $\delta\tilde{g}_{\mu\nu}$ (and not its derivatives) needs to be imposed to vanish on the boundary, when obtaining the Einstein field equations for $\tilde{g}_{\mu\nu}$. In the surface term, γ is the determinant of the induced metric on $\partial\mathcal{M}$ and $\tilde{K}$ is the trace of the extrinsic curvature of $\partial\mathcal{M}$.

The action for the vector field A_μ is given by

$$S_A = -\frac{1}{32\pi G}\int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\big[\kappa F_{\mu\nu}F^{\mu\nu} - 2\lambda\big(A_\mu A^\mu + 1\big)\big], \tag{6}$$

where $F_{\mu\nu} = \tilde{\nabla}_\mu A_\nu - \tilde{\nabla}_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$, κ is a dimensionless constant and λ is a Lagrange multiplier ensuring (2). The original action (6) has since been extended with three extra terms which are quadratic in $\tilde{\nabla}_\mu A_\nu$, see [15], in order to cure certain dynamical problems, see e.g. [11]. Here we consider the original TeVeS for simplicity. A detailed Hamiltonian analysis of the extended TeVeS model will be presented in a future communication.

The action for the scalar field ϕ is given by

$$S_\phi = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-\tilde{g}}\big[\mu\hat{g}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi + V(\mu)\big], \tag{7}$$

where μ is a non-dynamical dimensionless scalar field and $\hat{g}$ is a new metric defined as

$$\hat{g}^{\mu\nu} = \tilde{g}^{\mu\nu} - A^\mu A^\nu. \tag{8}$$

The potential term $V(\mu)$ is an arbitrary function that typically depends on a scale. The metric $\hat{g}^{\mu\nu}$ is used in the scalar field action, rather than $\tilde{g}^{\mu\nu}$, in order to avoid superluminal propagation of perturbations. For the same purpose, we assume that $\phi > 0$ [1].

All matter fields, denoted generically by χ^A, are coupled to the physical metric $g_{\mu\nu}$ so that their action has the form

$$S_m = \int_{\mathcal{M}} d^4x\sqrt{-g}\mathcal{L}\big[g, \chi^A, \nabla\chi^A\big]. \tag{9}$$

For simplicity, we will consider a scalar matter field χ with the action

$$S_m = -\int_{\mathcal{M}} d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + \mathcal{V}(\chi)\right]. \tag{10}$$

The determinant of the physical metric g is related to the determinant of the Bekenstein metric $\tilde{g}$ and the fields ϕ and A_μ as

$$g = e^{-4\phi}\big[1 - \big(1 - e^{-4\phi}\big)\big(A_\mu A^\mu + 1\big)\big]\tilde{g}. \tag{11}$$

3. Hamiltonian structure of TeVeS

The spacetime is assumed to admit a foliation into a union of nonintersecting spacelike hypersurfaces Σ_t, which are parameterized by the time t. The Bekenstein metric $\tilde{g}_{\mu\nu}$ induces a metric $h_{\mu\nu}$ on Σ_t, which is defined as

$$h_{\mu\nu} = \tilde{g}_{\mu\nu} + n_\mu n_\nu, \tag{12}$$

where n_μ is the future-directed unit normal to Σ_t. The unit normal can be written in terms of the lapse function N and the shift vector N^i $(i = 1, 2, 3)$ as

$$n_\mu = -\nabla_\mu t = (-N, 0, 0, 0), \qquad n^\mu = \left(\frac{1}{N}, -\frac{N^i}{N}\right). \tag{13}$$

Consequently, the metric $\tilde{g}_{\mu\nu}$ is decomposed in terms of the ADM variables h_{ij}, N and N^i as

$$\tilde{g}_{00} = -N^2 + N^i h_{ij} N^j, \qquad \tilde{g}_{0i} = h_{ij} N^j, \qquad \tilde{g}_{ij} = h_{ij}. \tag{14}$$

The determinant of the metric $\tilde{g}_{\mu\nu}$ is written as

$$\sqrt{-\tilde{g}} = N\sqrt{h}, \quad h = \det h_{ij}. \tag{15}$$

The vector field A_μ is decomposed into components tangent and normal to Σ_t as

$$_\perp A_\mu = h_\mu{}^\nu A_\nu, \qquad A_{\mathbf{n}} = n^\mu A_\mu, \tag{16}$$

respectively, where $h_\mu{}^\nu = h_{\mu\rho}\tilde{g}^{\rho\nu} = \delta_\mu{}^\nu + n_\mu n^\nu$ is the projection operator onto Σ_t. That is the components of the vector field are expressed as $A_0 = NA_{\mathbf{n}} + N^i A_i$ and $A_i = {}_\perp A_i$.

In the Hamiltonian formulation of TeVeS, the canonical momenta conjugate to h_{ij}, N, N^i, $A_{\mathbf{n}}$, A_i, λ, ϕ, μ and χ are denoted by π^{ij}, π_N, π_i, $p_{\mathbf{n}}$, p^i, p_λ, p_ϕ, p_μ and p_χ, respectively. Since the action is independent of the time derivatives of N, N^i, λ, $A_{\mathbf{n}}$ and μ, their canonically conjugated momenta are the primary constraints:

$$\pi_N \approx 0, \qquad \pi_i \approx 0, \qquad p_\lambda \approx 0, \qquad p_{\mathbf{n}} \approx 0, \qquad p_\mu \approx 0. \tag{17}$$

We obtain the total Hamiltonian in the form (with all primary constraints included through Lagrange multipliers)

$$\begin{aligned} H = \int_{\Sigma_t} d^3x \big(& N\mathcal{H}_T + N^i\mathcal{H}_i + v_N\pi_N + v^i\pi_i \\ & + v_\lambda p_\lambda + v_{\mathbf{n}} p_{\mathbf{n}} + v_\mu p_\mu \big) + H_{\mathrm{surf}}, \end{aligned} \tag{18}$$

where $\mathcal{H}_T$ is the Hamiltonian constraint, $\mathcal{H}_i$ is the momentum constraint, and H_{surf} is the surface contribution. The momentum constraint has the form

$$\begin{aligned} \mathcal{H}_i = & -2h_{ij}D_k\pi^{jk} - A_i\partial_j p^j + (\partial_i A_j - \partial_j A_i)p^j \\ & + \partial_i\phi p_\phi + \partial_i\chi p_\chi \approx 0, \end{aligned} \tag{19}$$

where D_k is the covariant derivative compatible with the metric h_{ij}. The momentum constraint defines the generator of the time-dependent spatial diffeomorphisms for the dynamical variables on Σ_t. The Hamiltonian constraint is responsible for the time evolution of the canonical variables. It consists of the contributions of the tensor, vector, scalar and matter fields,

$$\mathcal{H}_T = \mathcal{H}_T^{\mathrm{GR}} + \mathcal{H}_T^A + \mathcal{H}_T^\phi + \mathcal{H}_T^\chi \approx 0, \tag{20}$$

respectively. The tensor contribution is similar to GR,

$$\mathcal{H}_T^{\mathrm{GR}} = \frac{16\pi G}{\sqrt{h}}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl} - \frac{\sqrt{h}}{16\pi G}{}^{(3)}R, \tag{21}$$

where

$$\mathcal{G}_{ijkl} = \frac{1}{2}(h_{ik}h_{jl} + h_{il}h_{jk}) - \frac{1}{2}h_{ij}h_{kl} \tag{22}$$

and ${}^{(3)}R$ is the scalar curvature defined by the covariant derivative D_i. The contributions of the vector and scalar fields are

$$\begin{aligned} \mathcal{H}_T^A = & \frac{4\pi G}{\kappa\sqrt{h}}p^i h_{ij} p^j - A_{\mathbf{n}} D_i p^i \\ & + \frac{\kappa}{32\pi G}\sqrt{h}h^{ik}h^{jl}(D_iA_j - D_jA_i)(D_kA_l - D_lA_k) \\ & + \frac{\lambda}{16\pi G}\sqrt{h}\big(A_iA^i - A_{\mathbf{n}}^2 + 1\big) \end{aligned} \tag{23}$$

and

$$\begin{aligned} \mathcal{H}_T^\phi = & \frac{4\pi G}{\sqrt{h}\mu(1+A_{\mathbf{n}}^2)}p_\phi^2 + \frac{A_{\mathbf{n}}}{(1+A_{\mathbf{n}}^2)}p_\phi A^i\partial_i\phi \\ & - \frac{\mu\sqrt{h}}{16\pi G(1+A_{\mathbf{n}}^2)}\big(A^i\partial_i\phi\big)^2 \\ & + \frac{1}{16\pi G}\mu\sqrt{h}h^{ij}\partial_i\phi\partial_j\phi + V(\mu). \end{aligned} \tag{24}$$

The contribution of the matter field is the most interesting one, since it contains the contribution of the nonminimal coupling between the gravitational tensor, vector and scalar fields due to (1). The matter part of the Hamiltonian constraint is given as

$$\begin{aligned} \mathcal{H}_T^\chi = & \frac{\sqrt{1-(1-e^{-4\phi})\mathcal{G}_\lambda}}{2\sqrt{h}(e^{-4\phi} - (1-e^{-4\phi})A_iA^i)}p_\chi^2 \\ & - \frac{(1-e^{-4\phi})A_{\mathbf{n}}}{e^{-4\phi} - (1-e^{-4\phi})A_iA^i}A^i\partial_i\chi p_\chi \\ & + \sqrt{h\big(1-\big(1-e^{-4\phi}\big)\mathcal{G}_\lambda\big)} \times \\ & \times \Bigg[\frac{1-e^{-4\phi}}{2(e^{-4\phi} - (1-e^{-4\phi})A_iA^i)}\big(A^i\partial_i\chi\big)^2 \\ & + \frac{1}{2}h^{ij}\partial_i\chi\partial_j\chi + e^{-2\phi}\mathcal{V}(\chi)\Bigg]. \end{aligned} \tag{25}$$

Three more constraints are required in order to ensure that the primary constraints (17) are preserved in time,

$$\mathcal{G}_\lambda = A_iA^i - A_{\mathbf{n}}^2 + 1 \approx 0, \tag{26}$$

$$\mathcal{G}_{\mathbf{n}} = D_ip^i + \frac{\lambda\sqrt{h}}{8\pi G}A_{\mathbf{n}} + \ldots \approx 0, \tag{27}$$

$$\begin{aligned} \mathcal{G}_\mu = & \frac{4\pi G}{\sqrt{h}\mu^2(1+A_{\mathbf{n}}^2)}p_\phi^2 + \frac{\sqrt{h}}{16\pi G(1+A_{\mathbf{n}}^2)}\big(A^i\partial_ip_\phi\big)^2 \\ & - \sqrt{h}h^{ij}\partial_i\phi\partial_j\phi - \sqrt{h}\frac{\delta V(\mu)}{\delta\mu} \approx 0, \end{aligned} \tag{28}$$

where the constraint (27) has a complicated form, involving all the dynamical variables, and it has been omitted. The constraint (26) was introduced already in (25).

Note that the spatial part of the physical metric (1), namely $g_{ij} = e^{-2\phi}h_{ij} - 2\sinh(2\phi)A_iA_j$, changes its signature when the scalar field ϕ becomes larger than $\frac{1}{4}\ln(1+(A_iA^i)^{-1})$. This can be seen in the determinant

$$\det(g_{ij}) = e^{-2\phi}\big(e^{-4\phi} - \big(1-e^{-4\phi}\big)A_iA^i\big)h. \tag{29}$$

The change of signature is reflected in the matter part of the Hamiltonian constraint (25), where the denominator of the first three terms contains the same expression as (29), $e^{-4\phi} - (1 - e^{-4\phi})A_iA^i$. These terms diverge at $\phi = \frac{1}{4}\ln(1+(A_iA^i)^{-1})$ and change their signs thereafter. In particular, the kinetic term p_χ^2 becomes negative if ϕ is allowed to pass this point. In order to obtain a well-defined Hamiltonian formulation of matter in TeVeS, we require that the hypersurfaces Σ_t are spacelike in the physical frame. Combined with the requirement of no superluminal propagation of perturbations, $\phi > 0$, we obtain the restriction

$$0 < \phi < \frac{1}{4}\ln\left(1 + \frac{1}{A_iA^i}\right). \tag{30}$$

When the unit timelike vector field A_μ is dominated by the component $A_{\mathbf{n}}$, we have a weak spatial vector A_i, $0 \le A_iA^i \ll 1$, and hence the permitted region (30) for ϕ is large. Conversely, if

$A_iA^i \gg 1$, the permissible region for ϕ is narrow, with an upper limit of order $1/(4A_iA^i)$.

The first class constraints $\pi_N, \pi_i, \mathcal{H}_T, \mathcal{H}_i$ are associated with the invariance of the original theory under four-dimensional diffeomorphisms. The second class constraints $p_\lambda, p_{\mathbf{n}}, p_\mu, \mathcal{G}_\lambda, \mathcal{G}_{\mathbf{n}}, \mathcal{G}_\mu$ can be used to express the variables $\lambda, A_{\mathbf{n}}, \mu$ in terms of the gravitational variables h_{ij}, A_i, ϕ and the matter fields.

The surface term $H_{\rm surf}$ in the Hamiltonian (18) defines the total gravitational energy in space. The physical Hamiltonian is given by $H_{\rm phys} = H - H_{\rm b}$, where $H_{\rm b}$ is the Hamiltonian for a given reference background. We define the total energy associated with the time translation along $t^\mu = Nn^\mu + N^\mu$ for any given solution of the equations of motion as the value of the physical Hamiltonian when all the constraints are satisfied. For a stationary background, we obtain the total gravitational energy as

$$E = -\oint_{\partial\Sigma_t} d^2x\bigg(\frac{1}{8\pi G}N\sqrt{\sigma}\big({}^{(2)}K - {}^{(2)}K_{\rm b}\big) - 2N^ih_{ij}r_k\pi^{jk} - r_ip^i\big(NA_{\mathbf{n}} + N^jA_j\big)\bigg), \tag{31}$$

where σ, ${}^{(2)}K$ and r_i are the determinant of the induced metric, the trace of the extrinsic curvature and the unit normal for the boundary of Σ_t, respectively, ${}^{(2)}K_{\rm b}$ is the trace of the extrinsic curvature of the boundary on the reference background, and $A_{\mathbf{n}}$ is given by the constraint $\mathcal{G}_\lambda = 0$ as $A_{\mathbf{n}} = \pm\sqrt{A_iA^i + 1}$. The expression for the total energy (31) of TeVeS differs from that of GR in two ways: the metric on Σ_t is induced by $\tilde{g}_{\mu\nu}$ (not by $g_{\mu\nu}$) and the contribution of the vector field is included, namely the last term $\oint_{\partial\Sigma_t} d^2x r_ip^iA_0$. This generic expression for the total energy can be used to obtain the total energy with respect to different kinds of backgrounds, as in GR [16].

For an asymptotically flat spacetime, the expression (31) becomes the ADM energy of TeVeS. Recall that in GR the ADM energy satisfies the positive energy theorem [17,18]. We have not proven the positivity of the total energy (31) for an arbitrary isolated system, albeit we do expect that the positive energy theorem will hold in TeVeS. The ADM energy of the flat Minkowski spacetime is zero by definition. As a nontrivial example, we consider the spherically symmetric solution of the field equations of TeVeS with a vanishing radial vector component ($A^r = 0$) which was obtained in [19] using isotropic spherical coordinates as

$$\tilde{g}_{tt} = -\left(\frac{r-r_c}{r+r_c}\right)^{r_g/2r_c}, \qquad \tilde{g}_{rr} = \frac{(r^2-r_c^2)^2}{r^4}\left(\frac{r-r_c}{r+r_c}\right)^{-r_g/2r_c}, \tag{32}$$

where the characteristic radius is defined as

$$r_c = \frac{r_g}{4}\sqrt{1 + \frac{k}{\pi}\left(\frac{Gm_s}{r_g}\right)^2 - \frac{\kappa}{2}}, \tag{33}$$

and where the "scalar mass" m_s and the gravitational radius r_g are related to the total gravitational mass [1], and k is a dimensionless constant involved in the definition of the potential in the action (7). We obtain the ADM energy of the solution (32) as

$$E_{\rm ADM} = -\frac{1}{2G}\lim_{r\to\infty} r^2\frac{\partial h_{rr}}{\partial r} = \frac{r_g}{2G}. \tag{34}$$

The ADM energy depends on r_g rather than on the characteristic radius (33) of the solution. Identifying the ADM energy as the gravitational mass m of an isolated spherical matter distribution, one obtains $r_g = 2Gm$.

4. Discussion

We have uncovered the Hamiltonian structure of the original version of the TeVeS theory of gravity [1]. TeVeS is shown to contain six local gravitational degrees of freedom: two in the usual spin-2 graviton, three in the unit timelike vector field, and one in the scalar field. This is consistent with the previous knowledge on the theory. However, there is an important detail regarding the linearized theory. When we consider the lowest order perturbations in the absence of matter, with the background given as $\tilde{g}_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, $A^\mu = (1,0,0,0)$ and $\phi = \phi_c = \text{constant}$, the tensor perturbation has two traceless–transverse modes and the vector perturbation has two transverse modes, which all propagate at the uniform speed $e^{-2\phi_c}$ [11]. The scalar perturbation is a trace mode. The third degree of freedom which is associated with the vector field in the full nonlinear theory does not appear in the linearized theory. It appears, however, as an extra trace mode when the action of the vector field (6) is generalized (see [11]). Hence, in the case of the original version of TeVeS, the lowest order linearized theory lacks one degree of freedom.

The nonminimal coupling of the vector and scalar fields to the Bekenstein metric was found to be intricate, yet well defined. In the present Hamiltonian formulation, the nonminimal coupling is contained in the matter part (25) of the Hamiltonian constraint (20). The kinetic terms in the Hamiltonian constraint are positive definite, assuming (30) is satisfied, and hence there is no sign of ghost instability in TeVeS. This offers further support for the theoretical soundness of TeVeS. Complemented by the remarkable success of TeVeS in explaining the observed discrepancy between the dynamics and the distribution of the visible matter in galaxies, we can conclude that TeVeS is a highly interesting proposal for the extension of GR. There are further challenges and prospects.

It appears that some dark matter is still required in TeVeS, since otherwise TeVeS is unable to explain certain observations on galaxy clusters, gravitational lensing and the cosmic microwave background radiation [20,21]. It has been hypothesized that the required nonluminous matter could be composed of massive (sterile) neutrinos.

It is known that the Einstein–Hilbert action of GR (including the cosmological constant) is generated at one-loop order in any quantum field theory, when the geometry of the background spacetime is not fixed in the beginning [22–25]. This includes renormalizable higher-order derivative theories of gravity, such as Weyl gravity, where GR is induced at long distances. Obtaining TeVeS via such an induced mechanism is difficult, since matter couples to the physical metric. Hence the induced GR term is the physical curvature R, not the curvature $\tilde{R}$ defined by the Bekenstein metric. We speculate that the gravitational vector and scalar fields need to be present from the beginning and with nonminimal coupling to the background. Conceivably, in such a setting, the one-loop quantum corrections could generate the curvature part (5) of the TeVeS action. How else could the nonminimal coupling emerge?

These considerations do not address the quantum aspects of gravity itself. The quantization of TeVeS is indeed expected to be just as challenging as the quantization of GR.

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